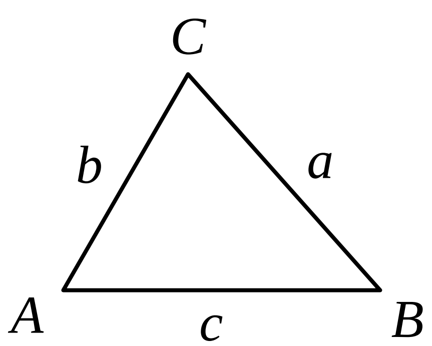
Law of Sines and Cosines

We have learned how to use trigonometry to solve right triangles. Now we will look at how trigonometry can help us solve oblique triangles.

Definition: An oblique triangle is one that does not contain a right angle.



****To solve an oblique triangle, we need to know at least one side and any two other parts of the triangle.**

We get the following 4 cases:

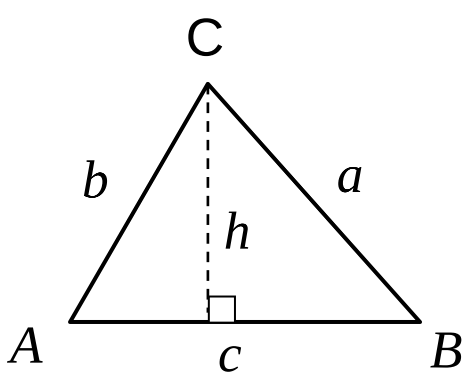
1. Two angles and any side (AAS or ASA)
2. Two sides and an angle opposite one of them (SSA)
3. Three sides (SSS)
4. Two sides and their included angle (SAS)

We use the Law of Sines for the first two cases.

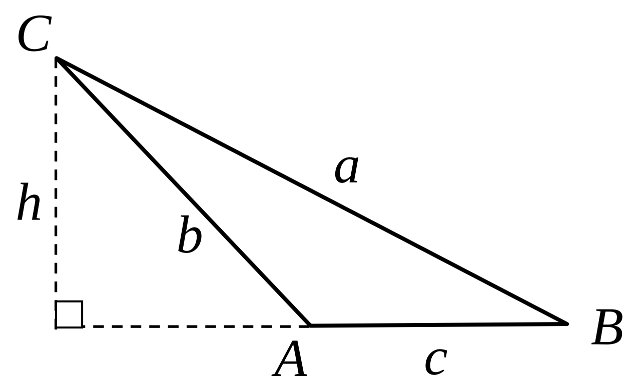
Law of Sines

If ABC is a triangle with sides a, b, and c, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{and} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

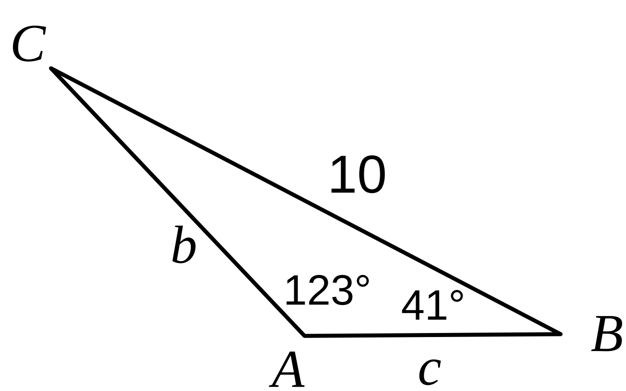


$\angle$ A is acute



$\angle$ A is obtuse

Example: Given A=123°, B=41° and a=10 inches, find c.



$\angle C$ = 180°-123°- 41° = 16°

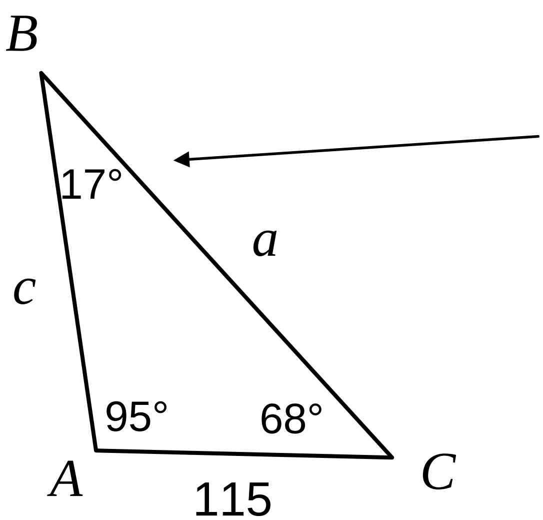
$$\frac{c}{\sin 16°} = \frac{10}{\sin 123°}$$

$$c = \frac{10 \sin 16°}{\sin 123°}$$

$$c \approx 3.29$$

**To use the Law of Sines you must always have one complete pair of angle and opposite side. Then you can solve for any of the other angles or sides.

Example: A triangular plot of land has interior angles $A = 95°$ and $C = 68°$. If the side between these angles is 115 yards long, what are the lengths of the other two sides?



$\angle B = 180° - 95° - 68° = 17°$

$$\frac{c}{\sin 68°} = \frac{115}{\sin 17°}$$

$$c = \frac{115 \sin 68°}{\sin 17°}$$

$$c \approx 364.69$$

$$\frac{a}{\sin 95°} = \frac{115}{\sin 17°}$$

$$a = \frac{115 \sin 95°}{\sin 17°}$$

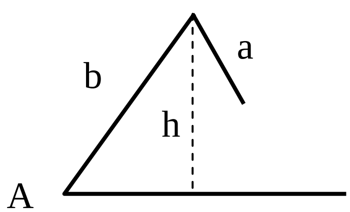
$$c \approx 391.84$$

The Ambiguous Case (SSA)

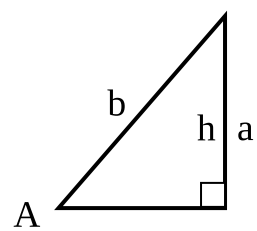
If <u>two sides</u> and <u>one opposite angle</u> are given, three possible situations can occur:

1. no such triangle exists
2. one such triangle exists
3. two distinct triangles may exist

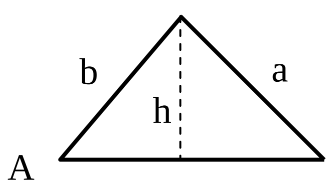
Consider a triangle with a, b, and A are given. ($h = b\sin A$)



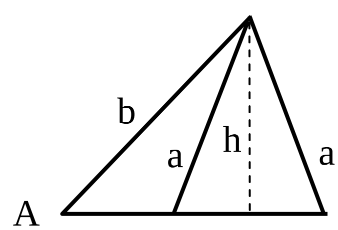
If A is acute, and a<h, then there is <u>no</u> triangle.



If A is acute, and a=h, then it is a right triangle and there is <u>one</u> triangle.

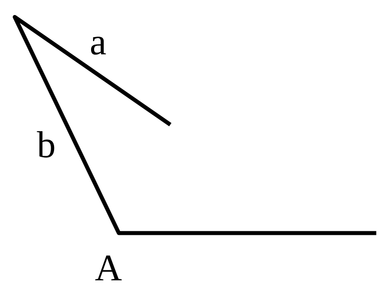


If A is acute, and a>b, then there is <u>one</u> triangle. (a > h also)

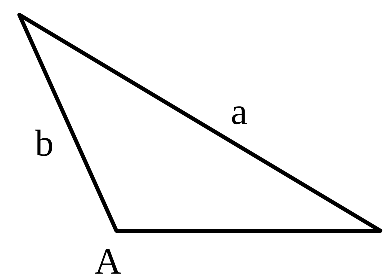


If A is acute, and h<a<b, then there are <u>two</u> triangles.

<u>Note</u>: If a < b, then find the height to determine the number of triangles.



If A is obtuse, and a#b, then there is no triangle.



If A is obtuse, and a>b, then there is one triangle.

Example: Given m$\angle A$ = 29°, a = 6, and b = 10, find m$\angle B$.

$$\frac{\sin B}{10} = \frac{\sin 29°}{6}$$

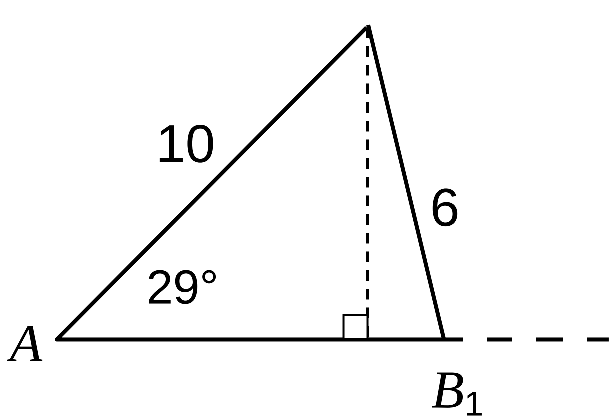
$$\sin B = \frac{10 \sin 29°}{6}$$

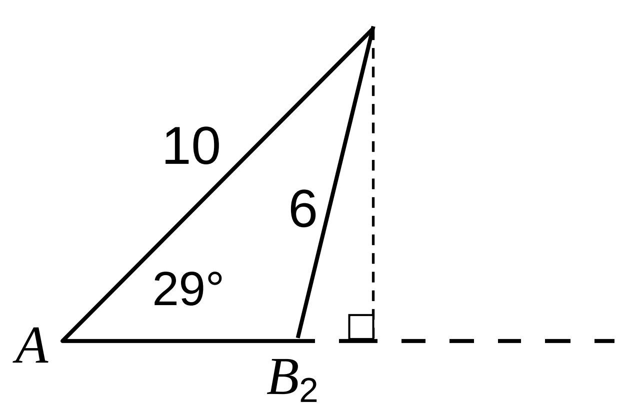
$$\sin B = .8080$$

$$B \approx 53.9°$$

Since $h = 10 sin 29° = 4.8$ and $4.8 < 6 < 10$, we must have 2 triangles.

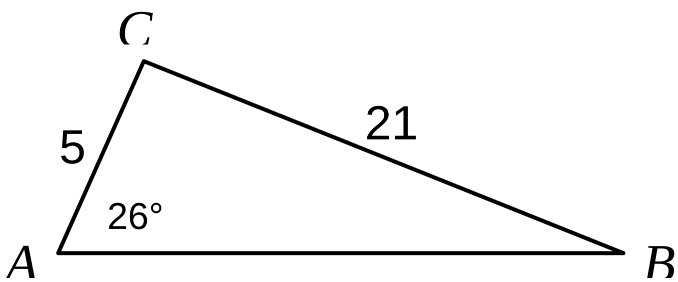
Using the calculator, we get $B \approx 54°$. Since this is an acute angle, let B_1=54° and find B_2. B_2 is a quadrant II angle with the same reference angle as B_1, so B_2 = $180° - 54° = 126°$.





Example: Given m$\angle A$ = 26°, b = 5 feet, and a = 21 feet, find the other side and the two other angles of the triangle.

Because the side across from the angle is longer than the adjacent side, we know there is one triangle.



$$\frac{\sin B}{5} = \frac{\sin 26°}{21}$$
$$\sin B = \frac{5\sin 26°}{21}$$
$$B = \sin^{-1}\left(\frac{5\sin 26°}{21}\right) \approx 6°$$

Since we know m$\angle A$ and m$\angle B$, we can find m$\angle C$.

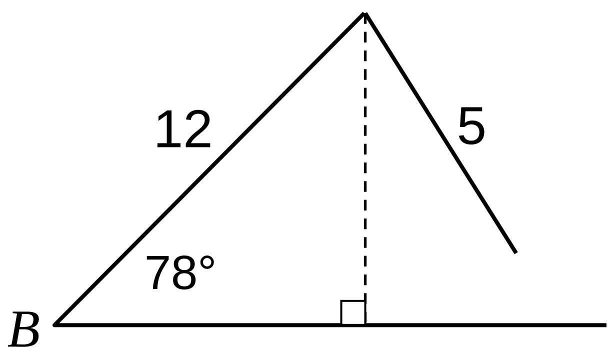
$$m\angle C = 180° - 26° - 6° = 148°$$

Use Law of Sines to find the length of side c.

$$\frac{c}{\sin 148°} = \frac{21}{\sin 26°}$$
$$c = \frac{21\sin 148°}{\sin 26°} \approx 25.4$$

Therefore, $c \approx$ 25.4 feet.

Example: Given m$\angle B$ = 78°, c = 12, and b = 5, find m$\angle C$.



$$h = 12 sin 78° = 11.7$$

Since $h > 5$, there is no triangle.

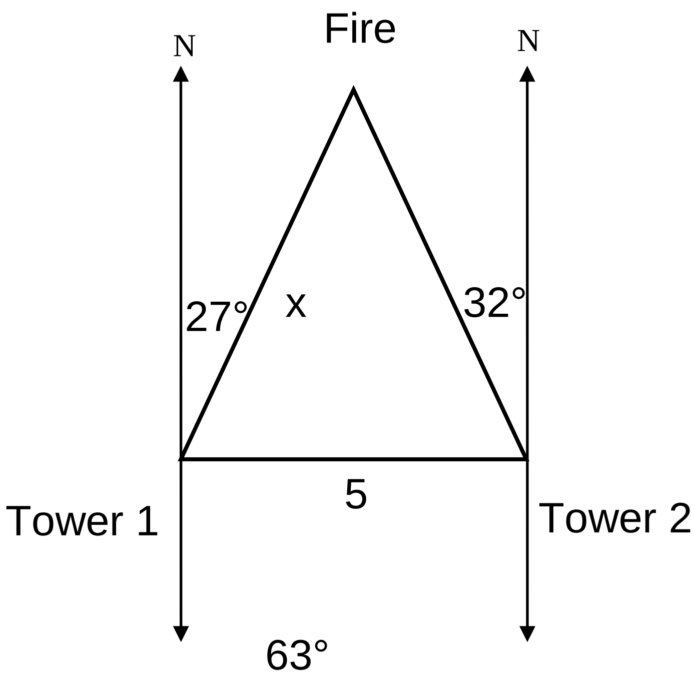
If you try to solve this triangle using the Law of Sines:

$$\frac{\sin 78°}{5} = \frac{\sin C}{12}$$

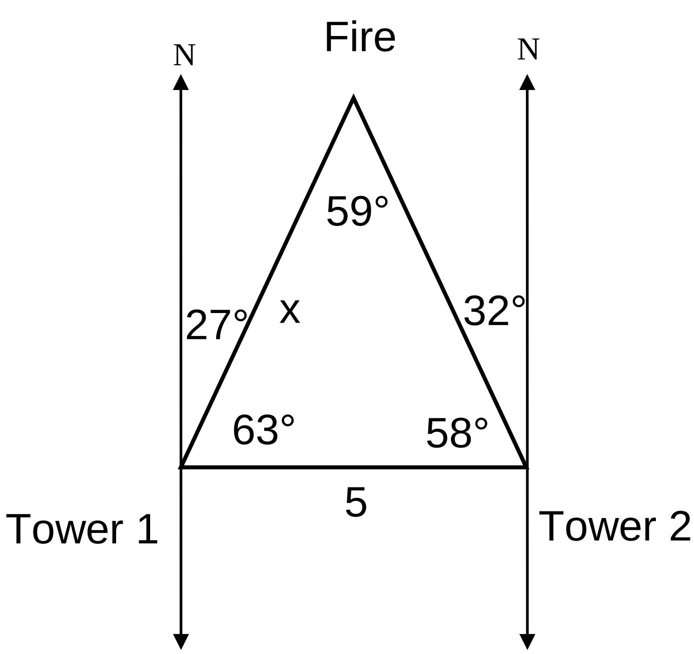
$$\sin C = \frac{12 \sin 78°}{5} \approx 2.35$$

Since the sine of an angle can never be greater than 1.0, this is impossible, and there is no such triangle with these measurements. (If you try to use [SIN] [x^{-1}] on your calculator, you will get an error message.)

Example: Two fire ranger towers lie on the east-west line and are 5 miles apart. There is a fire with a bearing of N 27°E from tower 1 and N 32° W from tower 2. How far is the fire from tower 1?



Since the tower is on an east-west line, it is perpendicular to the north-south line. So we can find the angle measurements of the lower 2 angles of the triangles, and thus the 3rd angle, also.



Use the Law of Sines to find x.

$$\frac{5}{\sin 59°} = \frac{x}{\sin 58°}$$

$$x = \frac{5\sin 58°}{\sin 59°}$$

$$x \approx 4.9 \text{ miles}$$

Law of Cosines

To solve an oblique triangle, we need to know at least one side and any two other parts of the triangle.

4 cases for oblique triangles:

1. Two angles and any side (AAS or ASA)
2. Two sides and an angle opposite one of them (SSA)
3. Three sides (SSS)
4. Two sides and their included angle (SAS)

We use the Law of Sines for the first two cases. For the last 2 cases we use the Law of Cosines.

Law of Cosines

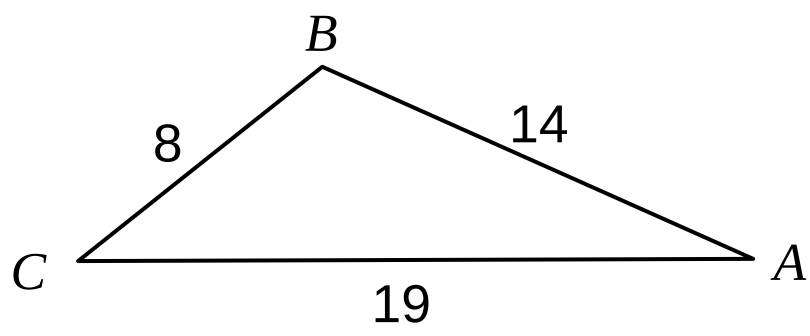
If ABC is a triangle with sides a, b, and c, then

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Example: Find the three angles of the triangle.



Find angle opposite the longest side first.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{8^2 + 14^2 - 19^2}{2(8)(14)}$$

$$\cos B \approx -0.45089$$

Since *cos B* is negative, we know by allsintancos that it is a 2nd quadrant angle, which means it is obtuse.

Using a calculator and $\cos^{-1}$, we find that $m\angle B \approx 116.8°$

You can use either the Law of Cosines or the Law of Sines to find the next angle.

$$\frac{\sin A}{8} = \frac{\sin 116.8°}{19}$$

$$\sin A = \frac{8 \sin 116.8°}{19} \approx 0.37583$$

If it weren't for the fact that we already have an obtuse angle in our triangle, we would have to consider that there are 2 angles that have sine=0.37583 (one acute and one obtuse). The acute angle is 22.08° and the obtuse angle is 180°- 22.08° = 157.92°.

Therefore, for our triangle, m$\angle A$=22.08°. Now find m$\angle C$.

$$m\angle C = 180° - 22.08° - 116.8 = 41.12°$$

***Note:** If the largest angle is obtuse, then the other 2 angles are acute. If the largest angle is acute, the other 2 angles will also be acute.

Example: Given C = 111°, a = 27, and b = 18, find the remaining side and two angles of the triangle.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 27^2 + 18^2 - 2(27)(18)\cos 111°$$

$$c^2 \approx 1401.33$$

$$c \approx 37.43$$

$$\frac{\sin A}{27} = \frac{\sin 111°}{37.43}$$

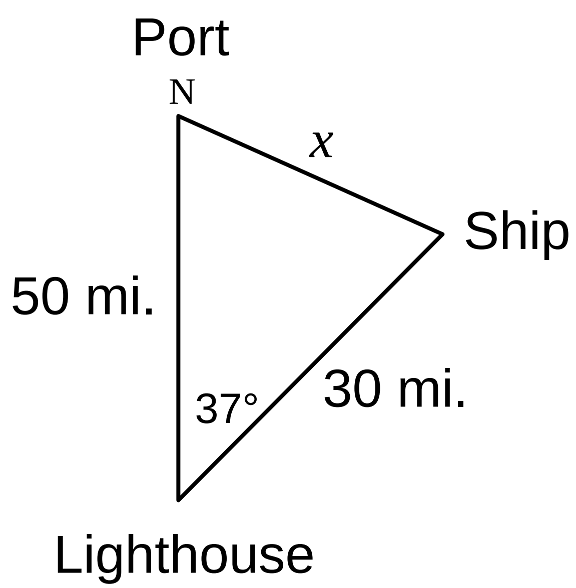
$$\sin A = \frac{27\sin 111°}{37.43}$$

$$A \approx 42.3°$$

$$B = 180° - 111° - 42.3° = 26.7°$$

Applications

Example: A port is 50 miles due north of a lighthouse. A ship is 30 miles from the lighthouse at a bearing of N 37° E. How far is the ship from the port?



$$x^2 = 50^2 + 30^2 - 2(50)(30)\cos 37°$$

$$x^2 \approx 1004.09$$

$$x \approx 31.69 \text{ miles}$$