Applications of Matrices and Determinants

Cramer's Rule

If a system of n linear equation in n variables has a coefficient matrix A with a nonzero determinant $|A|$, the solution of the system is

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \ldots, \ x_n = \frac{|A_n|}{|A|}$$

where the ith column of A_i is the column of constants in the system of equations. If the determinant of the coefficient matrix is zero, the system has either no solution or infinitely many solutions.

Example: Look at $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$

Coefficient Matrix	D	D_x	D_y
$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$

Example: Solve $\begin{cases} 2x - 3y = 7 \\ 5x - 2y = -3 \end{cases}$ using Cramer's Rule.

$$x = \frac{\begin{vmatrix} 7 & -3 \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 5 & -2 \end{vmatrix}} = \frac{-14 - 9}{-4 - (-15)} = \frac{-23}{11}$$

$$y = \frac{\begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 5 & -2 \end{vmatrix}} = \frac{-6 - 35}{11} = \frac{-41}{11}$$

Example: Solve the system $\begin{cases} 2x + y + z = 6 \\ -x - y + 3z = 1 \\ y - 2z = -3 \end{cases}$

$$x = \frac{\begin{vmatrix} 6 & 1 & 1 \\ 1 & -1 & 3 \\ -3 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & 3 \\ 0 & 1 & -2 \end{vmatrix}} = \frac{(12-9+1)-(3+18-2)}{(4+0-1)-(0+6+2)} = \frac{4-19}{3-8} = \frac{-15}{-5} = 3$$

$$y = \frac{\begin{vmatrix} 2 & 6 & 1 \\ -1 & 1 & 3 \\ 0 & -3 & -2 \end{vmatrix}}{-5} = \frac{(-4+0+3)-(0-18+12)}{-5} = \frac{-1+6}{-5} = \frac{5}{-5} = -1$$

$$z = \frac{\begin{vmatrix} 2 & 1 & 6 \\ -1 & -1 & 1 \\ 0 & 1 & -3 \end{vmatrix}}{-5} = \frac{(6+0-6)-(0+2+3)}{-5} = \frac{0-5}{-5} = \frac{-5}{-5} = 1$$

The solution is (3, -1, 1).

Area of a Triangle

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the symbol ± indicates that the appropriate sign should be chosen to yield a positive area.

Example: Find the area of a triangle whose vertices are (-3, 1), (2, 4), and (5, - 3).

$$Area = \pm\frac{1}{2}\begin{vmatrix} -3 & 1 & 1 \\ 2 & 4 & 1 \\ 5 & -3 & 1 \end{vmatrix} = \pm\frac{1}{2}\left(-13-31\right) = \pm\frac{1}{2}(-44) = 22$$

Example: Find a value of x such that the triangle with the vertices (-4, 2), (0, 2) and (-2, y) has an area of 4 square units.

$$4 = \pm\frac{1}{2}\begin{vmatrix} -4 & 2 & 1 \\ 0 & 2 & 1 \\ -2 & y & 1 \end{vmatrix}$$

$$\pm 8 = \begin{vmatrix} -4 & 2 & 1 \\ 0 & 2 & 1 \\ -2 & y & 1 \end{vmatrix}$$

$$\pm 8 = (-8 - 4) - (-4 - 4y)$$

$$\pm 8 = -12 + 4 + 4y$$

$$\pm 8 + 8 = 4y$$

$$y = \frac{8 \pm 8}{4}$$

$$y = \frac{16}{4} \text{ or } \frac{0}{4}$$

$$y = 4 \text{ or } 0$$

One point is (-2, 4) and another is (-2, 0).

Test for Collinear Points

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear (lie on the same line) if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example: Determine whether the points (- 2, 4), (0, 3), and (8, - 1) are collinear.

$$\begin{vmatrix} -2 & 4 & 1 \\ 0 & 3 & 1 \\ 8 & -1 & 1 \end{vmatrix} = (-6 + 32) - (24 + 2) = 26 - 26 = 0$$

Because the determinant = 0, the points are collinear.

Example: Determine whether the points (3,2), (1, -2), and (5, - 1) are collinear.

$$\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & 1 \\ 5 & -1 & 1 \end{vmatrix} = (-6+10-1)-(-10-3+2) = 3+11 = 14$$

The points are not collinear.

<u>Two-Point Form of the Equation of a Line</u>

An equation of the line passing through the distinct points (x_1, y_1), (x_2, y_2) is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Example: Find the equation of the line through the points (- 2, 9) and (3, - 1).

$$\begin{vmatrix} x & y & 1 \\ -2 & 9 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$$(9x + 3y + 2) - (27 - x - 2y) = 0$$

$$10x + 5y - 25 = 0$$

$$5y = -10x + 25$$

$$y = -2x + 5$$

Example: Find the equation of the line through the points (0, 3) and (1, - 5).

$$\begin{vmatrix} x & y & 1 \\ 0 & 3 & 1 \\ 1 & -5 & 1 \end{vmatrix} = 0$$

$$(3x + y) - (3 - 5x) = 0$$

$$8x + y - 3 = 0$$

$$y = -8x + 3$$