Sequences and Series

You can think of a sequence as a function whose domain is the set of positive integers.

$$f(1)=a_1, \quad f(2)=a_2, \quad f(3)=a_3,\ldots f(n)=a_n,\ldots$$

Definition of Sequence

An infinite sequence is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4,\ldots a_n,\ldots$$

are the terms of the sequence. If the domain of the function consists of the first n positive integers only, the sequence is a finite sequence.

. **Example**: Write the first four terms of the following sequences.

a) $a_n = 2n+5$

$a_1 = 2(1) + 5 = 7$
$a_2 = 2(2) + 5 = 9$
$a_3 = 2(3) + 5 = 11$
$a_4 = 2(4) + 5 = 13$

b) $b_n = 3^{n-1}$

$b_1 = 3^{1-1} = 3^0 = 1$
$b_2 = 3^{2-1} = 3^1 = 3$
$b_3 = 3^{3-1} = 3^2 = 9$
$b_4 = 3^{4-1} = 3^3 = 27$

c) $c_n = \dfrac{(-1)^n}{n^2+1}$

$$c_1 = \frac{(-1)^1}{1^2+1} = \frac{-1}{2}$$

$$c_2 = \frac{(-1)^2}{2^2+1} = \frac{1}{5}$$

$$c_3 = \frac{(-1)^3}{3^2+1} = \frac{-1}{10}$$

Notice how the signs alternate.

$$c_4 = \frac{(-1)^4}{4^2+1} = \frac{1}{17}$$

$$c_5 = \frac{(-1)^5}{5^2+1} = \frac{-1}{26}$$

Because different sequences can have the first terms the same, it is necessary to know the *n*th term in order to define a unique sequence.

The *n*th term can be thought of as the "rule" for the sequence.

Finding the *n*th Term of a Sequence

To find the apparent pattern, list the terms and underneath list the numbers for *n*. Look for a pattern that shows what is done to *n* to get the term in the pattern.

Example: Write an expression for the apparent *n*th term (a_n) of the sequence 1, 3, 5, 7,…

n:	1	2	3	4	…	*n*
Terms:	1	3	5	7	…	a_n

Look at how the term can be arrived at using the given *n* value. It appears that we can double the *n* value and then subtract 1. This tells us that $a_n = 2n - 1$.

Example: Write an expression for the apparent *n*th term (a_n) of the sequence 2, -4, 6, -8, 10,…

n:	1	2	3	4	5	…	*n*
Terms:	2	-4	6	-8	10	…	a_n

Solution: $a_n = (-1)^{n+1}(2n)$

Recursive Sequences

To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms.

Example: 1, 1, 2, 3, 5, 8, 13, 21, 34, ….

To get the next term, we add the previous 2 terms. But in order to define the sequence, we have to list the first 2 terms, because there are no 2 terms "before" those terms that we can add. The sequence is defined as:

$$a_1 = 1,\ \ a_2 = 1,\ a_n = a_{n-2} + a_{n-1}$$

This is a well known sequence called the Fibonacci Sequence.

Factorial Notation

When we want to multiply a product such as

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

we use factorial notation.

Definition: If n is a positive integer, n factorial is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdot \cdot (n-1) \cdot n$$

As a special case, 0! = 1.

Example: Evaluate 6!

Solution: $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

Example: Write the first 4 terms of the sequence defined by

$$a_n = \frac{n^2}{n!}$$

$$a_1 = \frac{1^2}{1!} = 1, \quad a_2 = \frac{2^2}{2!} = 2, \quad a_3 = \frac{3^2}{3!} = \frac{9}{6} = \frac{3}{2}, \quad a_4 = \frac{4^2}{4!} = \frac{16}{24} = \frac{2}{3}$$

Note: $2n! = 2(1 \bullet 2 \bullet 3 \bullet 4 \bullet \bullet \bullet \bullet\ n)$
$(2n)! = 1 \bullet 2 \bullet 3 \bullet 4 \bullet \bullet \bullet \bullet 2n$

Evaluating Factorial Expressions

Example: Evaluate the factorial expressions.

a) $\dfrac{10!}{2! \cdot 8!}$

Expanding, we get: $\dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}$.

After canceling, we end up with: $\dfrac{10 \cdot 9}{2 \cdot 1} = \dfrac{5 \cdot 9}{1} = 45$

You should be able to go from $\dfrac{10!}{2! \cdot 8!}$ to $\dfrac{10 \cdot 9}{2}$.

b) $\dfrac{2! \cdot 6!}{3! \cdot 5!}$

solution: $\dfrac{2! \cdot 6!}{3! \cdot 5!} = \dfrac{2! \cdot 6}{3!} = \dfrac{6}{3} = 2$

c) $\dfrac{n!}{(n+1)!}$

solution: $\dfrac{n!}{(n+1)!} = \dfrac{n!}{(n+1)n!} = \dfrac{1}{n+1}$

<u>Factorials Using a Graphing Calculator</u>

- The factorial key function can be accessed by [MATH] [PRB] [!].

Example: Find 15! using your calculator.

Press 15 [MATH] [PRB] [!] [ENTER].

Solution: The screen will show the answer . $1.3 \text{ x}10^{12}$

Example: Find 7!-6! using your calculator.

Press 7 [MATH] [PRB] [!] - 6 [MATH] [PRB] [!] [ENTER].

Solution: 4320

Graphing Sequences on a Graphing Calculator

1. Put the calculator in Sequence mode by pressing [MODE] and then on the 4th line, highlight "Seq." Press [ENTER] and then [2nd] [QUIT].

2. Enter the sequence in the [Y=] screen. Your sequence can be named u, v, or w. We will use u for these instructions.
   - Set the nMin = 1 to show that your values for n will start with the number 1.
   - Type in the formula for your nth term after u(n) =. (You can use the [X,T,θ,n] key for n.)
   - You do not have to enter the first term, u(nMin) unless your sequence is recursive.

3. Move your cursor to the very left of the u(n) =. By repeatedly pressing [ENTER] you can choose how the graph will be displayed. Select the dotted option.

4. Press [GRAPH]. Then press [ZOOM] [ZoomFit] for the best viewing window.

   - Note: You can also change the viewing window from the [WINDOW] screen. You will see that you can set the min and max for n, as well as for x and y.

Example: Graph the first 10 terms of the sequence defined by $u_n = 2 - \frac{4}{n}$

Example: Graph the first 10 terms of the sequence defined by $v_n = (-1)^{n+1}(2n)$

<u>To View Identify Individual Terms Using a Graphing Calculator:</u>

1. Using the [TRACE] feature:

   - Press [TRACE] and then the UP ARROW once. You should see displayed on your screen the values of n and x (which are identical) and y (which is the value of the term). Use the right and left arrows to move from term to term.

2. Using the [TABLE] feature:

   - Press [TBLSET]. Set the following values:

     TblStart = 1 (since n starts at 1).
     ΔTbl = 1 (since the n values go up by 1's).
     Indpnt: Auto
     Depend: Auto

   - Press [TABLE] to see the terms listed.

Finding the *n*th Term Using a Graphing Calculator

Your calculator has a key for each of the 3 sequences u, v, and w. The keys are the [2nd] function of the 7, 8, and 9 keys respectively. To find the nth term of a sequence, do the following:

1. Enter the sequence at the [Y=] screen.
2. Press [[2nd] [QUIT].
3. Press u (using [2nd] [7]). The u should appear on your screen.
4. Type in the number of the term you want, enclosed by parentheses and press [ENTER].

Example: Use Find the 5th term of the sequence

$$u_n = 2 - \frac{4}{n}$$

using the following methods:

1. Direct calculation of the term.
2. The [TRACE] feature on the graph of the sequence.
3. The [TABLE] feature.
4. Entering u(n) directly into the calculator.

Solution: $u_5 = 1.2$

Summation Notation

We often want to find the sum of the terms of a finite sequence. The notation we use is called summation notation or sigma notation because it involves the Greek letter sigma, written as Σ.

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$$

where i is called the index of summation, n is the upper limit of summation, and 1 is the lower limit of summation.

Example: $\sum_{i=1}^{5} 4i = 4(1) + 4(2) + 4(3) + 4(4) + 4(5) = 60$

Example: Find each sum.

a) $\sum_{i=1}^{4} 2i - 1$

$$\sum_{i=1}^{4} 2i - 1 = 1 + 3 + 5 + 7 = 16$$

b) $\sum_{i=1}^{5} (-1)^{i-1} i!$

$$\sum_{i=1}^{5} (-1)^{i-1} i! = (-1)^0(1!) + (-1)^1(2!) + (-1)^2(3!) + (-1)^3(4!) + (-1)^4(5!)$$
$$= 1 + (-2) + 6 + (-24) + 120$$
$$= 101$$

c) $\sum_{k=3}^{5} (1 + k^2)$

$$\sum_{k=3}^{5} (1 + k^2) = 10 + 17 + 26 = 53$$

*Note: The lower limit does not have to be 1 and the index does not have to be i.

Properties of Sums

1. $\sum_{i=1}^{n} c = cn$, c is a constant.

2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, c is a constant.

3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Series

Many applications involve the sum of the terms of a finite or infinite sequence. Such a sum is called a series.

Definition of Series

Consider the infinite sequence $a_1, a_2, a_3, a_4, \ldots a_n, \ldots$

1. The sum of the first n terms of the sequence is called a finite series or the nth partial sum of the sequence and is denoted by

$$a_1 + a_2 + a_3 + a_4 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

2. The sum of all terms of the infinite sequence is called an infinite series and is denoted by

$$a_1 + a_2 + a_3 + a_4 + \ldots + a_i + \ldots = \sum_{i=1}^{\infty} a_i$$

Example: Consider the series $\sum_{i=1}^{\infty} \frac{3}{10^i}$.

a) Find the 3rd partial sum.

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} = 0.3 + 0.03 + 0.003 = 0.333$$

b) Find the sum of the infinite series.

$$\sum_{i=1}^{\infty}\frac{3}{10^i}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10{,}000}+\ldots$$
$$=0.3+0.03+0.003+0.0003+\ldots$$
$$=0.3333\ldots$$
$$=\frac{1}{3}$$

<u>Finding Partial Sums on a Graphing Calculator</u>

You can use the [sum] feature along with the [seq(] feature to find partial sums of a sequence. The function [sum( ] is used to find the partial sum of a sequence. The format for the [seq(] command are:

seq(*expression, variable, begin, end, increment*)

(The default for increment is 1, so if your increment is 1, you do not have to type it in.)

To find the kth partial sum of the sequence a_n, do the following:

- Press [2nd][LIST] [MATH][sum( ] [2nd][LIST] [OPS][seq( ] $a_n, n, 1, k$)) [ENTER].

If you have the sequence already entered at the [Y=] screen, then you can do the following:

- Press [2nd][LIST] [MATH][sum( ] [2nd][LIST] [OPS][seq( ] u, $n, 1, k$)) [ENTER].

If you want a list of the first k partial sums, do the following:

- Press [2nd][LIST] [OPS][cumSum( ] [2nd][LIST] [OPS][seq( ] $a_n, n, 1, k$)) [ENTER].

The first k partial sums will be listed as a set in { }.

If you have the sequence already entered at the [Y=] screen, then you can do the following:

- Press [2nd][LIST] [OPS][cumSum( ] [2nd][LIST] [OPS][seq( ] u, $n, 1, k$)) [ENTER].

Example: Find the 6th partial sum of the sequence defined by $a_n = 2^n$.

Solution: Press [2nd][LIST] [MATH][sum( ] [2nd][LIST] [OPS][seq( ] 2^n, n, 1, 6)) [ENTER].

answer = 126

Example: Find the sum $\sum_{i=1}^{4} \frac{3}{10^i}$ by first entering the sequence into the [Y=] screen.

Solution: After entering the sequence into the sequence u, press Press [2nd][LIST] [MATH][sum( ] [2nd][LIST] [OPS][seq( ] u, n, 1, 4)) [ENTER].

answer = 0.3333

Example: Find the first 5 partial sums of the series $\sum_{i=1}^{5} 2^n$

Solution: Press [2nd][LIST] [OPS][cumSum( ] [2nd][LIST] [OPS][seq( ] 2^n, n, 1, 5)) [ENTER].

answer = {2, 6, 14, 30, 62}

Application

Example: If a deposit of $50 is made each month into an account that earns 6% interest compounded monthly, then the balance in the account after n months is

$$A_n = 50(200)[(1.005)^n - 1].$$

Find the balance in the account after 5 years.

Solution: 5 years is 60 months, so $n = 60$. Find A_{60}.

$$A_{60} = 50(200)[(1.005)^{60} - 1] = \$3488.50$$